On limits of sequences of hyperarithmetical functionals and predicates

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Introduction. Let N be the set of all natural numbers and N^N the set of all 1-place number-theoretic functions. In [1], Addison mentions that there is a close analogy between the finite Borel hierarchy and the Kleene's arithmetical hierarchy. In this paper, we shall obtain an effective counterpart of the Baire-de la Vallée Poussin's hierarchy for the classical (B)-measurable sets and functions, especially in the case of transfinite orders. To state our results, we shall explane some notations and definitions used here. But ones in Kleene [3, 4 and 5] will be used often without explanation. Let α and x denote finite sequences of function-variables $\alpha_1, \cdots, \alpha_r$ (each running over N^N) and of number-variables $x_1, \cdots, x_s$ (each running over N), respectively. We shall construct below a primitive recursive function $\omega^r(a)$ (for a given positive integer r) such that if $a \in O$ then the following (0.1)–(0.3) hold for all α (i.e., for all $\alpha_1, \cdots, \alpha_r$):

$$\omega^r(a) \in O^\alpha , \tag{0.1}$$

$$|\omega^r(a)|^\alpha = |a| , \tag{0.2}$$

(0.3) For any natural number y if $y <_O a$ then $\omega^r(y) <_{O^\alpha} \omega^r(a)$.

For $r=0$, let $\omega^r(a)=a$. For each $a \in O$ we shall define the classes $\mathcal{K}_a$ and $\mathfrak{M}_a$ of predicates $P(\alpha, x)$, where $r \geq 0$, $s \geq 0$ and $r+s>0$, as follows:

For $a=1$, $\mathcal{K}_1$ is the class of all general recursive predicates $P(\alpha, x)$.

For $a=2^b \neq 1$ (such a is denoted by b^*), $\mathcal{K}_a$ consists of all predicates $P(\alpha, x)$ such that for some predicate $R(\alpha, x, n)$ contained in $\mathcal{K}_b$

$$P(\alpha, x) \equiv \lim_n R(\alpha, x, n)$$

holds, that is,

$$P(\alpha, x) \equiv (Ek)(n)_{n \geq k} R(\alpha, x, n) \equiv (k)(En)_{n \geq k} R(\alpha, x, n) . \tag{0.4}$$

For $a=3 \cdot 5^b$, $\mathcal{K}_a$ consists of all predicates $P(\alpha, x)$ such that there are a predicate $R(\alpha, x, n)$ and general recursive functions $\xi(n)$, $\eta(n)$ for which (0.4) and the following (0.5), (0.6) hold:

$$\xi(n) <_O a \text{ for all } n , \tag{0.5}$$

(0.6) For each n, $\lambda x R(\alpha, x, n) \leq_T [\eta(n)]\{H^{\alpha}_{\omega^{r}(\xi(n))}, \alpha\}$[1] for all α, where H^{α}_{y} is the H-predicate of Kleene.

Remark 1. As is seen later, such predicates $\lambda\alpha x R(\alpha, x, n)(n=0,1,\cdots)$ belong to $\cup\{\mathcal{K}_c : c <_0 a\}$.

For each $a \in O$, $\mathfrak{M}_a$ is the class of all predicates $P(\alpha, x)$ such that there exist general recursive functions $\phi(x)$, $\phi'(x)$ by which

$$P(\alpha, x) \equiv H^{\alpha}_{\omega^{r}(a)^*}(\phi(x)) \equiv \overline{H^{\alpha}_{\omega^{r}(a)^*}}(\phi'(x))$$

is satisfied. The following lemma can be proved by the method of Kleene [5].

Lemma 1. *Let $a \in O$ and $P(\alpha, x)$ be a given predicate. If*

$$P(\alpha, x) \equiv (Ek)(n)R_0(\alpha, x, n) \equiv (k)(En)R_1(\alpha, x, n) \tag{0.7}$$

holds for some predicates R_0, R_1 contained in $\mathfrak{M}_a$, then $P(\alpha, x)$ belongs to the class $\mathfrak{M}_{a^*}$; and vice versa.

Let $d(n)=n_0$ (i.e., $d(0)=1$ and $d(n+1)=d(n)^*$). The fact that for every n $\mathcal{K}_{d(n)}=\mathfrak{M}_{d(n)}$ is known by Addison [1] (using above Lemma 1) and also by Kondô [6] with respect to somewhat different hierarchy. Its proof, e.g., is seen in [9, Lemma 3]. In this paper, we shall prove that for each $a \in O$ $\mathcal{K}_a=\mathfrak{M}_{a^*}$ if $\overline{Fin}(a)$ (i.e., if the ordinal $|a|$ represented by a is infinite). Similar definitions and results for functionals (including number-theoretic functions) can be obtained, too. (See § 4 below.)

On the other hand, Tugué and the author jointly prove the following theorem [10; Theorem 4]:

Theorem A. *The class $\cup\{\mathfrak{M}_a : a \in O\}$ coincides with the family of all Δ^1_1-predicates.*

Since results mentioned above give the classifications of the hyperarithmetical predicates and functionals by Theorem A, they are the effective versions of the corresponding facts of the classical theory of (B)-measurable sets and functions. Further we shall obtain (in § 5 below) the effective versions of the separation theorems for the classical Borelean sets in Baire's space.

Remark 2. All the results stated in this paper can be of course relativized with respect to a given function $\xi \in N^N$.

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[1] According to Enderton [2], this shall mean that "for each n, $\lambda x R(\alpha, x, n)$ is recursive in $H^{\alpha}_{\omega^{r}(\xi(n))}, \alpha$ with Gödel number $\eta(n)$." Namely

$$R(\alpha, x, n) \equiv U(\mu y T_s^{H^{\alpha}_{\omega^{r}(\xi(n))}, \alpha}(\eta(n), x, y))=0.$$

§1. **Lemma 2.** *Let r be a given positive integer. Then there is a primitive recursive function $\omega^r(a)$ such that if $a \in O$ then the following* (1.1)–(1.3) *hold for all α:*

(1.1) $$\omega^r(a) \in O^\alpha ,$$

(1.2) $$| \omega^r(a) |^\alpha = | a | ,$$

(1.3) *For any y, if $y <_O a$ then $\omega^r(y) <_{O^\alpha} \omega^r(a)$.*

Proof. Let q be a solution of the following recursion equation:

$$\{z\}(y) \simeq \begin{cases} y & \text{if } y=1 , \\ \{z\}((y)_0)^* & \text{if } y=(y)_0^* \neq 1 , \\ 3\cdot 5^u & \text{if } y=3\cdot 5^{(y)_2} , \\ 0 & \text{otherwise} , \end{cases}$$

where $u = S_1^{2,\overbrace{1,\cdots,1}^{r}}(p, z, y)$ and p is a uniform Gödel number of $\lambda zyt\{z\}(\{(y)_2\}(t))$ from α. And we define the function $\omega^r(y)$ as follows:

$$\omega^r(y) \simeq \{q\}(y).$$

Then this function is general recursive, but by Kleene's method [4] we can show that it is primitive recursive. Now let $M(a)$ denote the predicate obtained from the conjunction of (1.1)–(1.3) by prefixing the universal quantifier (α). We shall prove that if $a \in O$ then $M(a)$.

Case 1. $a=1$. This case is obvious.

Case 2. $a=2^b \neq 1$. By the hypothesis of induction we have $M(b)$. Namely,

(1.1′) $$(\alpha)[\omega(b) \in O^\alpha] ,$$

(1.2′) $$(\alpha)[| \omega(b) |^\alpha = | b |] ,$$

(1.3″) $$(\alpha)(y)[y <_O b \rightarrow \omega(y) <_{O^\alpha} \omega(b)] ,$$

where we abbreviate $\omega^r(a)$ by $\omega(a)$. Since $\omega(a) = \omega(b)^*$, $\omega(a) \in O^\alpha$ for all α by (1.1′). Next by (1.2′) we have:

$$| \omega(a) |^\alpha = | \omega(b) |^\alpha + 1 = | b | + 1 = | a | .$$

Finally suppose $y <_O a$. Then $y \leq_O b$, and hence by (1.3′) we have

$$\omega(y) \leq_{O^\alpha} \omega(b) <_{O^\alpha} \omega(b)^* = \omega(a) .$$

Case 3. $a = 3\cdot 5^b \in O$. As $\{b\}(d(n)) <_O a$ for all n, by the hypothesis of induction $M(\{b\}(d(n)))$ for all n. Hence for any n and α we have:

(1.1″) $$\omega(\{b\}(d(n))) \in O^\alpha ,$$

(1.2″) $$| \omega(\{b\}(d(n))) |^\alpha = | \{b\}(d(n)) | ,$$

(1.3″) if $y <_O \{b\}(d(n))$ then $\omega(y) <_{O^\alpha} \omega(\{b\}(d(n))) <_{O^\alpha} \omega(\{b\}(d(n+1)))$.

Since $\omega(a)=3\cdot 5^u$, where $u=S_1^{2,\overbrace{1,\cdots,1}^{r}}(p,q,a)$, and $\{(\omega(a))_2\}^\alpha(d(n))=\omega(\{b\}(d(n)))$, it holds

$$\{(\omega(a))_2\}^\alpha(d(n))\in O^\alpha\ ,$$

$$|\{(\omega(a))_2\}^\alpha(d(n))|^\alpha=|\{b\}(d(n))|\ ,$$

and
$$\{(\omega(a))_2\}^\alpha(d(n))<_{O^\alpha}\{(\omega(a))_2\}^\alpha(d(n+1))\ .$$

By the definition of O^α, we have $\omega(a)\in O^\alpha$ and $|\omega(a)|^\alpha=|a|$. To prove (1.3), suppose $y<_O a$. Then there is an n such that $y<_O\{b\}(d(n))$. By (1.3″) and the definition of $<_{O^\alpha}$ it holds:

$$\omega(y)<_{O^\alpha}\omega(\{b\}(d(n)))=\{(\omega(a))_2\}^\alpha(d(n))<_{O^\alpha}\omega(a)\ .$$

This completes the proof of Lemma 2.

For simplicity, we shall hereafter write ω for ω^r.

Lemma 3. *Let $a,b\in O$. If $|a|=|b|$ then $\mathfrak{M}_a=\mathfrak{M}_b$.*

Proof. Let $P(\alpha,x)$ be in the class $\mathfrak{M}_a$. Then there are recursive functions $\phi(x),\phi'(x)$ such that $P(\alpha,x)\equiv H^\alpha_{\omega(a)*}(\phi(x))$ and $\bar{P}(\alpha,x)\equiv H^\alpha_{\omega(a)*}(\phi'(x))$. Hence we have

$$\begin{aligned}P(\alpha,x)&\equiv(Ey)T_1^{H^\alpha_{\omega(a)},\alpha}(\phi(x),\phi(x),y)\\&\equiv(Ey)T_1^{1,1}(\overline{\{\lambda tU(\mu zT_1^{H^\alpha_{\omega(b)},\alpha}(\theta^1(\omega(a),\omega(b)),t,z))\}(y)},\bar{\alpha}(y),\phi(x),\phi(x))\ ,\end{aligned}$$

with the primitive recursive function θ^1 defined in Spector [8, Thm. 5], since $|\omega(a)|^\alpha=|\omega(b)|^\alpha$. Hence by Kleene [5, Lemma 1] we can find a recursive function $\psi(x)$ such that

$$P(\alpha,x)\equiv(Ey)T_1^{H^\alpha_{\omega(b)},\alpha}(\psi(x),\psi(x),y)\equiv H^\alpha_{\omega(b)*}(\psi(x))\ .$$

Similarly we can obtain $\bar{P}(\alpha,x)=H^\alpha_{\omega(b)*}(\psi'(x))$ for some recursive function $\psi'(x)$. Consequently P belongs to the class $\mathfrak{M}_b$. Hence we have $\mathfrak{M}_a=\mathfrak{M}_b$.

Lemma 4. (Addison [1]) Let $a\in O$. *If $Fin(a)$, then $\mathcal{K}_a=\mathfrak{M}_a$.*

By Lemma 1 $P(\alpha,x)$ is in $\mathfrak{M}_{d(n)}$ if and only if it is expressible in the both $(n+1)$-quantifier forms (i,e., it is a Δ^0_{n+1}-predicate). Hence the lemma is just [9; Lemma 3].

§2. Lemma 5. *Let $a\in O$. If $\overline{Fin}(a)$ then $\mathcal{K}_a\subseteq\mathfrak{M}_{a^*}$.*

Proof. Let $P(\alpha,x)$ be in the class $\mathcal{K}_a$.

Case 1. $a=3\cdot 5^b$. Then there are a predicate R and general recursive functions ξ,η such that (0.4)–(0.6) hold. Since

$$H^\alpha_{\omega(\xi(n))}\leq_T[\rho^1(\omega(\xi(n)),\omega(a))]\{H^\alpha_{\omega(a)},\alpha\}$$

with the primitive recursive function ρ^1 defined in Kleene [5; § 6.6][2], we have the following equivalences:

[2] To be exact, ρ^1 should be written as $\rho^{\overbrace{1,\cdots,1}^{r}}$.

$$R(\alpha, x, n) \equiv U(\mu y T_s^{H^\alpha_{\omega(\xi(n))}, \alpha}(\eta(n), x, y)) = 0 ,$$

$$\equiv U(\mu y T_s^{1,1}(\overline{\{\lambda t U(\mu z T_1^{H^\alpha_{\omega(a)}, \alpha}(\rho^*(n), t, z))\}}(y), \bar{\alpha}(y), \eta(n), x)) = 0 ,$$

where $\rho^*(n) = \rho^1(\omega(\xi(n)), \omega(a))$,

$$\equiv (Ey) T_1^{H^\alpha_{\omega(a)}, \alpha}(\varphi(x, n), \varphi(x, n), y) \equiv H^\alpha_{\omega(a)*}(\varphi(x, n)) ,$$

$$\equiv (y) \bar{T}_1^{H^\alpha_{\omega(a)}, \alpha}(\varphi'(x, n), \varphi'(x, n), y) \equiv \bar{H}^\alpha_{\omega(a)*}(\varphi'(x, n)) ,$$

for suitable primitive recursive functions φ and φ' by Kleene [5; Lemma 1]. Hence the predicate R belongs to the class $\mathfrak{M}_a$. The predicates

$$\text{(2.1)} \qquad \text{“}n \geq k \rightarrow R(\alpha, x, n)\text{” and “}n \geq k \ \& \ R(\alpha, x, n)\text{”}$$

are also in the class $\mathfrak{M}_a$. Consequently, P belongs to the class $\mathfrak{M}_{a^*}$, by Lemma 1 and (0.4).

Case 2. $a = 2^b \neq 1$. Then there is a predicate $R(\alpha, x, n)$ contained in $\mathcal{K}_b$ such that (0.4) holds. By the hypothesis of induction, R is contained in $\mathfrak{M}_{b^*} = \mathfrak{M}_a$. Hence the predicates (2.1) also are in the class $\mathfrak{M}_a$. Thus by Lemma 1 and (0.4), P belongs to the class $\mathfrak{M}_{a^*}$. (Q.E.D.)

Let $a \in O$ and $a \neq 1$. We say that $P(\alpha, x)$ is contained in the class $\mathcal{B}_a$, when the following conditions are satisfied:

(i) If $a = b^*$, there are predicates $R_i(\alpha, x, n)$ in $\mathcal{K}_b (i = 0, 1)$ such that

$$\text{(2.2)} \qquad P(\alpha, x) \equiv (En) R_0(\alpha, x, n) \equiv (n) R_1(\alpha, x, n) .$$

(ii) If $a = 3 \cdot 5^b$, then there are predicates $R_i(\alpha, x, n)$ and general recursive functions ξ_i, η_i $(i = 0, 1)$ such that (2.2) and the following (2.3), (2.4) hold for $i = 0, 1$:

$$\text{(2.3)} \qquad \xi_i(n) <_O a \text{ for all } n,$$

$$\text{(2.4)} \qquad \text{For each } n, \ \lambda x R_i(\alpha, x, n) \leq_T [\eta_i(n)] \{H^\alpha_{\omega(\xi(n))}, \alpha\} \text{ for all } \alpha.$$

Then we have the

Lemma 6. *Let $a \in O$ and $|a|$ be a limit number. Then we have $\mathfrak{M}_a = \mathcal{B}_a$.*

Proof. Let $\omega(a) = 3.5^b$. Suppose $P(\alpha, x)$ belongs to the class $\mathfrak{M}_a$. Then there are general recursive functions φ_0, φ_1 such that

$$P(\alpha, x) \equiv H^\alpha_{\omega(a)*}(\varphi_0(x)) \text{ and } \bar{P}(\alpha, x) \equiv H^\alpha_{\omega(a)*}(\varphi_1(x)) .$$

Now let the functional $F\langle\alpha\rangle(y)$ defined as follows:

$$\text{(2.5)} \qquad F\langle\alpha\rangle(y) = \begin{cases} 0 & \text{if } H^\alpha_{\omega(a)}(y) (\equiv H^\alpha_{b^\alpha_{(y)_1}}((y)_0)) , \\ 1 & \text{otherwise} , \end{cases}$$

where $b^\alpha_z = \{b\}^\alpha(d(z))$. We shall define the predicates $R_i(\alpha, x, n)$ $(i = 0, 1)$ as follows:

$$\text{(2.6)} \qquad R_i(\alpha, x, n) \equiv (Ey)_{y < n+1} T_1^{1,1}(\overline{F\langle\alpha\rangle}(y), \bar{\alpha}(y), \varphi_i(x), \varphi_i(x)) .$$

Since $P^{(i)}(\alpha, x) \equiv (Ey)T_1^{1,1}(\overline{F\langle\alpha\rangle}(y), \bar{\alpha}(y), \varphi_i(x), \varphi_i(x))$, where $P^{(0)}$ and $P^{(1)}$ denote P and $\bar{P}$, respectively, we have

$$(2.7) \qquad P^{(i)}(\alpha, x) \equiv (En)R_i(\alpha, x, n) \qquad (i=0, 1) .$$

This is (2.2). Now we shall find general recursive functions ξ_i, η_i by which (2.3) and (2.4) are satisfied. Let ξ_i define as follows:

$$\xi_0(n) = \xi_1(n) = \{(a)_2\}(d(\mathrm{Max}\,\{(y)_1 : y < n+1\})) .$$

We shall write simply ξ for this function. Obviously ξ is general recursive and

$$(2.8) \qquad \xi(n) <_O a \quad \text{for all } n .$$

Using the primitive recursive function ρ^1 defined in Kleene [5; § 6.6][2] we obtain

$$(2.9) \qquad H^\alpha_{b^\alpha_{(y)_1}} \leq_T [\rho^1(b^\alpha_{(y)_1}, \omega(\xi(n)))]\{H^\alpha_{\omega(\xi(n))}, \alpha\} \quad \text{for } y < n+1 ,$$

since $b^\alpha_z = \omega(\{(a)_2\}(d(z)))$. Hence the function

$$\sigma(y, n) = \rho^1(\omega(\{(a)_2\}(d((y)_1))), \omega(\xi(n)))$$

is general recursive; and by (2.5), (2.6), (2.9) we have

(2.10)

$$R_i(\alpha, x, n)$$
$$\equiv (Ey)_{y<n+1} T_1^{1,1}(\overline{\{\lambda t\, U(\mu z\, T_1^{H^\alpha_{\omega(\xi(n))}, \alpha}(\sigma(y, n), t, z))\}}(y), \bar{\alpha}(y), \varphi_i(x), \varphi_i(x))$$

Let $u_i (i=0, 1)$ be a uniform Gödel number of the predicate (of x, n), obtained from the righthand member of (2.10) by replacing $H^\alpha_{\omega(\xi(n))}$ by A, from A and α. Then it holds:

$$(2.11) \qquad \lambda x R_i(\alpha, x, n) \leq_T [S_s^{\overbrace{1,1,1,\cdots,1}^{r}}(u_i, n)]\{H^\alpha_{\omega(\xi(n))}, \alpha\}$$

for all n, α and for $i=0, 1$. Take $\eta_i(n) = S_s^{\overbrace{1,1,1,\cdots,1}^{r}}(u_i, n)$. Then $\xi_i (=\xi)$ and $\eta_i (i=0, 1)$ are the desired functions, by (2.8) and (2.11). (Futher, since $R_i(\alpha, x, n) \rightarrow R_i(\alpha, x, n+1)$, we have $P^{(i)}(\alpha, x) \equiv \lim_n R_i(\alpha, x, n)$ by (2.7). Consequently, P belongs to the class $\mathcal{K}_a$.)

Conversely, suppose that for a predicate $P(\alpha, x)$ there exist predicates R_i and general recursive functions ξ_i, $\eta_i (i=0, 1)$ such that (2.2)–(2.4) hold. First we shall show that R_i's belong to the class $\mathfrak{M}_a$: By (2.4)

$$R_i(\alpha, x, n) \equiv U(\mu y\, T_s^{H^\alpha_{\omega(\xi_i(n))}, \alpha}(\eta_i(n), x, y)) = 0 .$$

Since $H^\alpha_{\omega(\xi_i(n))} \leq_T [\rho_i(n)]\{H^\alpha_{\omega(a)}, \alpha\}$, where $\rho_i(n) = \rho^1(\omega(\xi_i(n)), \omega(a))$, we obtain

$$R_i(\alpha, x, n) \equiv U(\mu y\, T_s^{1,1}(\overline{\{\lambda t\, U(\mu z\, T_1^{H^\alpha_{\omega(a)}, \alpha}(\rho_i(n), t, z))\}}(y), \bar{\alpha}(y), \eta_i(n), x) = 0 .$$

Hence we can find primitive recursive functions σ_i, σ'_i such that

$$R_i(\alpha, x, n) \equiv H^\alpha_{\omega(a)*}(\sigma_i(x, n)) \equiv \overline{H^\alpha_{\omega(a)*}}(\sigma'_i(x, n)) .$$

From this and (2.2) using the method of proof of Lemma 1, it follows that P is contained in $\mathfrak{M}_a$. Thus the proof of Lemma 6 has completed.

For example, the predicate H_{a^*} (where $|a|$ is a limit number) is in the class $\mathcal{K}_a$. Its proof is similar to the first-half of the proof of Lemma 6. But obviously H_{a^*} is not contained in $\mathfrak{M}_a$. Hence we have the

Corollary 1. *For $a \in O$, if $|a|$ is a limit number, then* $\mathfrak{M}_a \subsetneqq \mathcal{K}_a$.

Of course this corollary is also an immediate consequence of Theorem 1 below.

Remark 3. If $\mathfrak{M}$ is a class of predicates, then $\mathfrak{M}^*$ denotes the class of all predicates $P(\alpha, x)$ such that for some predicate R contained in $\mathfrak{M}$

$$P(\alpha, x) \equiv \lim_n R(\alpha, x, n) .$$

Then for each $a \in O$ we have

$$(\mathcal{K}_a)^* = \mathcal{K}_{a^*} \quad \text{and} \quad (\mathfrak{M}_a)^* = \mathfrak{M}_{a^*} .$$

The former is just the definition of $\mathcal{K}_{a^*}$. The latter will be proved as follows: $(\mathfrak{M}_a)^* \subseteq \mathfrak{M}_{a^*}$ is proved by the same method as in Case 2 of the proof of Lemma 5. Conversely, suppose $P(\alpha, x)$ belongs to $\mathfrak{M}_{a^*}$. By Lemma 1 there are predicates R_0, R_1 contained in $\mathfrak{M}_a$ such that (0.7) holds. By the same method as in the proof of [9; Lemma 3] we can find a predicate R contained in $\mathfrak{M}_a$ such that $P(\alpha, x) \equiv \lim_n R(\alpha, x, n)$ holds.[3] Hence P belongs to the class $(\mathfrak{M}_a)^*$.

§ 3. **Lemma 7.** *Let $a \in O$ and $|a|$ be a limit number. If $P(\alpha, x) \equiv \lim_n Q(\alpha, x, n)$ for some Q contained in $\mathfrak{M}_a$, then P belongs to the class $\mathcal{K}_a$. Namely $(\mathfrak{M}_a)^* \subseteq \mathcal{K}_a$.*

Proof. Since Q is contained in $\mathfrak{M}_a$, by Lemma 6 we have Q_i, ξ (ξ, η_i are general recursive) for $i=0, 1$ such that

$$\xi(m) <_O a \text{ for all } m , \tag{3.1}$$

$$\lambda x n Q_i(\alpha, x, n, m) \leq_T [\eta_i(m)]\{H^{\alpha}_{\omega(\xi(m))}, \alpha\} \tag{3.2}$$

for all m, α,

$$Q(\alpha, x, n) \equiv (m) Q_0(\alpha, x, n, m) \equiv (Em) Q_1(\alpha, x, n, m) . \tag{3.3}$$

By the hypothesis we have

$$\begin{aligned} P(\alpha, x) &\equiv (Ek)(n)[(n)_0 \geq k \rightarrow Q_0(\alpha, x, (n)_0, (n)_1)] \\ &\equiv (k)(En)[(n)_0 \geq k \;\&\; Q_1(\alpha, x, (n)_0, (n)_1)] \end{aligned}$$

[3] Take $R(\alpha, x, n)$ as follows:

$$\begin{aligned} &(Ek)_{k \leq n}[(i)_{i<n} R_0(\alpha, x, k, i) \;\&\; (Ei)_{i<n}(j)_{j<n} \bar{R}_0(\alpha, x, j, (i)_j) \\ &\quad \& \;\; (i)_{i \leq k}\{(Em)_{m<n} R_1(\alpha, x, i, m) \vee (m)_{m<n}(Ej)_{j<i} \bar{R}_1(\alpha, x, j, (m)_j)\}] . \end{aligned}$$

As in the proof of [9; Lemma 3] we shall define R as follows:

$$R(\alpha, x, n) \equiv (Ek)_{k\leq n}[(i)_{i<n}\{(i)_0 \geq k \rightarrow Q_0(\alpha, x, (i)_0, (i)_1)\}$$
$$\& \quad (Ei)_{i<n}(j)_{j<k}\{(i)_{j,0} \geq j \quad \& \quad \bar{Q}_0(\alpha, x, (i)_{j,0}, (i)_{j,1})\}$$
$$\& \quad (m)_{m\leq k}\{(Ei)_{i<n}[(i)_0 \geq m \quad \& \quad Q_1(\alpha, x, (i)_0, (i)_1)]$$
$$\vee (i)_{i<n}(Ej)_{j<m}[(i)_{j,0} \geq j \rightarrow \bar{Q}_1(\alpha, x, (i)_{j,0}, (i)_{j,1})]\}] \, .$$

(Cf. Footnote 3).)
Then it can be proved that

$$(3.4) \qquad P(\alpha, x) \equiv \lim_n R(\alpha, x, n) \, .$$

By advancing the quantifiers in the definition of R, we have the following equivalence:

$$R(\alpha, x, n) \equiv (Ek)_{k\leq n}(Eh)_{h<n}(m)_{m<n}(l)_{l<k}(p)_{p\leq k}(i)_{i<n}(Ej)_{j<p}(Eq)_{q<n}$$
$$[\{(m)_0 \geq k \rightarrow Q_0(\alpha, x, (m)_0, (m)_1)\} \quad \& \quad \{(h)_{l,0} \geq l \ \& \ \bar{Q}_0(\alpha, x, (h)_{l,0}, (h)_{l,1})\}$$
$$\& \quad \{[(q)_0 \geq p \ \& \ Q_1(\alpha, x, (q)_0, (q)_1)] \vee [(i)_{j,0} \geq j \rightarrow \bar{Q}_1(\alpha, x, (i)_{j,0}, (i)_{j,1})]\}] \, .$$

By (3.2) we have

$$Q_0(\alpha, x, (m)_0, (m)_1) \equiv U(\mu y T_{s+1}^{H_{u_1}^\alpha, \alpha}(\eta_0((m)_1), x, (m)_0, y)) = 0 \, ,$$
$$\bar{Q}_0(\alpha, x, (h)_{l,0}, (h)_{l,1}) \equiv U(\mu y T_{s+1}^{H_{u_2}^\alpha, \alpha}(\eta_0((h)_{l,1}), x, (h)_{l,0}, y)) = 1 \, ,$$
$$Q_1(\alpha, x, (q)_0, (q)_1) \equiv U(\mu y T_{s+1}^{H_{u_3}^\alpha, \alpha}(\eta_1((q)_1), x, (q)_0, y)) = 0$$

and

$$\bar{Q}_1(\alpha, x, (i)_{j,0}, (i)_{j,1}) \equiv U(\mu y T_{s+1}^{H_{u_4}^\alpha, \alpha}(\eta_1((i)_{j,1}), x, (i)_{j,0}, y)) = 1 \, ,$$

where $u_1 = \omega(\xi((m)_1))$, $u_2 = \omega(\xi((h)_{l,1}))$, $u_3 = \omega(\xi((q)_1))$ and $u_4 = \omega(\xi((i)_{j,1}))$. Since $m<n$, $h<n$ & $l<k\leq n$, $q<n$ and $i<n$ & $j<p\leq k\leq n$, we define the function ν as follows:

$$\nu(n) = \operatorname*{Max}_{<_O} \{\xi((m)_1), \xi((h)_{l,1}) \mid m, h, l<n\} \, ,$$

where $\operatorname*{Max}_{<_O}$ denotes the Maximum with respect to $<_O$. As $\xi(k) <_O a$ for all k, $\lambda kn[\xi(k) \leq_O \xi(n)]$ is general recursive by Kleene [4; § 21]. Hence ν is also general recursive; and for $m<n$, $h<n$ & $l<n$, $q<n$, $i<n$ & $j<n$ the following four relations hold:

$$H_{u_1}^\alpha \leq_T [\rho_1(m, n)]\{H_u^\alpha, \alpha\}$$
$$H_{u_2}^\alpha \leq_T [\rho_2(h, l, n)]\{H_u^\alpha, \alpha\} \, ,$$
$$H_{u_3}^\alpha \leq_T [\rho_1(q, n)]\{H_u^\alpha, \alpha\} \, ,$$

and

$$H_{u_4}^\alpha \leq_T [\rho_2(i, j, n)]\{H_u^\alpha, \alpha\} \, ,$$

where $u = \omega(\nu(n))$; $\rho_1(m, n) = \rho^1(u_1, u)$, $\rho_2(h, l, n) = \rho^1(u_2, u)$. So, it holds:

$$
\begin{aligned}
(3.5)\quad R(\alpha, x, n) \equiv {} & (Ek)_{k\leq n}(El)_{l<n}(m)_{m<n}(l)_{l<k}(p)_{p\leq k}(i)_{i<n}(Ej)_{i<p}(Eq)_{q<n} \\
& [\{(m)_0 \geq k \rightarrow U(\mu y T_{s+1}^{1,1}(\overline{\{\lambda t U(\mu z T_1^{H_u^\alpha,\alpha}(\rho_1(m, n), t, z))\}}(y), \bar{\alpha}(y), \\
& \qquad \eta_0((m)_1), x, (m)_0)) = 0\} \\
& \& \ \{(h)_{l,0} \geq l \ \& \ U(\mu y T_{s+1}^{1,1}(\overline{\{\lambda t U(\mu z T_1^{H_u^\alpha,\alpha}(\rho_2(h, l, m), t, z))\}}(y), \bar{\alpha}(y), \\
& \qquad \eta_0((h)_{l,1}), x, (h)_{l,0})) = 1\} \\
& \& \ \{[(q)_0 \geq p \ \& \ U(\mu y T_{s+1}^{1,1}(\overline{\{\lambda t U(\mu z T_1^{H_u^\alpha,\alpha}(\rho_1(q, n), t, z))\}}(y), \bar{\alpha}(y), \\
& \qquad \eta_1((q)_1), x, (q)_0)) = 0] \\
& \vee [(i)_{j,0} \geq j \rightarrow U(\mu y T_{s+1}^{1,1}(\overline{\{\lambda t U(\mu z T_1^{H_u^\alpha,\alpha}(\rho_2(i, j, n), t, z))\}}(y), \bar{\alpha}(y), \\
& \qquad \eta_1((i)_{j,1}), x, (i)_{j,0})) = 1]\}] .
\end{aligned}
$$

Let w be a uniform Gödel number of the predicate (of x, n), obtained from the righthand member of (3.5) by replacing H_u^α by A, from A and α. Then we have

$$R(\alpha, x, n) \equiv \{S_s^{\overbrace{1,1,1,\cdots,1}^{r}}(w, n)\}^{H_u^\alpha,\alpha}(x) = 0 ,$$

namely, for each n

$$(3.6) \qquad \lambda x R(\alpha, x, n) \leq_T [\sigma(n)]\{H_{\omega(\nu(n))}^\alpha, \alpha\} \quad \text{for all } \alpha ,$$

where $\sigma(n) = S_s^{1,1,1,\cdots,1}(w, n)$. Since $\nu(n) <_0 a$ and the function $\sigma(n)$ is primitive recursive, it follows that P belongs to the class $\mathcal{K}_a$, by (3.4) and (3.6). This completes the proof of Lemma 7.

By this lemma together with Lemma 5 and Remark 3 we have: $\mathfrak{M}_{a^*} = \mathcal{K}_a$ if $a \in O$ and $|a|$ is limit. Thus, by an induction with respect to the definition of $a \in O$ using Remark 3 again, we have the

Theorem 1. *For each $a \in O$, if $\overline{Fin}(a)$, then $\mathfrak{M}_{a^*} = \mathcal{K}_a$.*

Corollary 2. *Let $a, b \in O$. If $|a| = |b|$ then $\mathcal{K}_a = \mathcal{K}_b$.*

This follows directly from Lemma 3 and Theorem 1.

Of course, if $a, b \in O$ and $|a| < |b|$ then $\mathcal{K}_a \subsetneqq \mathcal{K}_b$ and $\mathfrak{M}_a \subsetneqq \mathfrak{M}_b$. By Theorem A stated in the introduction, this theorem together with Lemma 4 is the effective counterpart of the classical fact mentioned in Ljapunow et al [7; II in p. 33]. So, we obtain the effective hyperborelean hierarchy.

Corollary 3. *Let $a \in O$. Then $\mathcal{B}_{a^*} = \mathfrak{M}_a = \mathcal{K}_a$ if $Fin(a)$; and $\mathcal{B}_a = \mathfrak{M}_a$ if $\overline{Fin}(a)$.*

Proof. Let $Fin(a)$ and $P(\alpha, x)$ be a predicate contained in $\mathcal{B}_{a^*}$. Then there are predicates $R_i (i = 0, 1)$ in $\mathcal{K}_a$ such that

$$(3.7) \qquad P(\alpha, x) \equiv (En) R_0(\alpha, x, n) \equiv (n) R_1(\alpha, x, n) .$$

By Lemma 4 R_i's belong to $\mathfrak{M}_a$, and hence P also belongs to $\mathfrak{M}_a$. So $\mathcal{B}_{a^*} \subseteq \mathfrak{M}_a$. Obviously $\mathcal{K}_a \subseteq \mathcal{B}_{a^*}$. Hence by Lemma 4, $\mathcal{B}_{a^*} = \mathfrak{M}_a = \mathcal{K}_a$.

Next let $\overline{Fin}(a)$. If $a=3\cdot 5^b$ then this Corollary is just Lemma 6. Now suppose $a=b^*$. If P belongs to $\mathcal{B}_a=\mathcal{B}_{b^*}$, then there are R_i in $\mathcal{K}_b$ such that (3.7) holds. By Theorem 1 R_i's belong to $\mathfrak{M}_a$. Hence P is contained in $\mathfrak{M}_a$. So, $\mathcal{B}_a\subseteq\mathfrak{M}_a$. On the other hand $\mathfrak{M}_a=\mathcal{K}_b\subseteq\mathcal{B}_{b^*}=\mathcal{B}_a$. Consequently $\mathcal{B}_a=\mathfrak{M}_a$. (Q.E.D)

By Corollaries 1 and 3, it is seen that the classes $\mathcal{B}_a$ ($a\in O$ and $a\neq 1$) are the effective counterparts of the classical $B_\gamma(\gamma<\Omega)$ in the descriptive set theory. (See Ljapunow et al. [7; pp. 14–16].)

Further we can obtain the following propositions which correspond to some classical facts [7; p. 12]:

Proposition 1. *Let $a\in O$ and $|a|$ be a limit number. If $P(\alpha, x)$ is in the class $\mathcal{K}_a$ then P can be expressed in the following form:*

$$P(\alpha, x)\equiv(En)Q(\alpha, x, n)\ , \tag{3.8}$$

where Q is a predicate such that

$$Q(\alpha, x, n)\rightarrow\bar{Q}(\alpha, x, m) \qquad \text{if } n\neq m\ , \tag{3.9}$$

and such that there are Q_1, ξ, η (ξ, η are general recursive functions) by which the following (3.10)–(3.12) *are satisfied:*

$$\xi(i)<_{O}a \qquad \text{for all } i, \tag{3.10}$$

$$\text{For all } i, \alpha \quad \lambda xnQ_1(\alpha, x, n, i)\leq_T[\eta(i)]\{H^{\alpha}_{\omega(\xi(i))}, \alpha\}\ . \tag{3.11}$$

$$Q(\alpha, x, n)\equiv(i)Q_1(\alpha, x, n, i)\ . \tag{3.12}$$

Proof. As P is contained in $\mathcal{K}_a$ and $|a|$ is a limit number, there are R and recursive ξ', η' such that

$$\xi'(n)<_{O}a \qquad \text{for all } n\ , \tag{3.13}$$

$$\lambda xR(\alpha, x, n)\leq_T[\eta'(n)]\{H^{\alpha}_{\omega(\xi'(n))}, \alpha\}\ , \tag{3.14}$$

$$P(\alpha, x)\equiv\lim_n R(\alpha, x, n)\ . \tag{3.15}$$

Then we have

$$P(\alpha, x)\equiv(En)[(i)_{i<n}\bar{R}(\alpha, x, i)\ \&\ (i)R(\alpha, x, n+i)]\ .$$[4]

We shall define $Q_1(\alpha, x, n, i)$ as follows:

$$Q_1(\alpha, x, n, i)\equiv(j)_{j<i}[\{j\geq n\vee\bar{R}(\alpha, x, j)\}\ \&\ \{j<n\vee R(\alpha, x, j)\}]\ ,$$

and let $Q(\alpha, x, n)\equiv(i)Q_1(\alpha, x, n, i)$. For this Q, (3.8) and (3.9) hold. Further one can find such recursive functions ξ, η. (Q is also contained in the class $\mathcal{K}_a$.) (Q.E.D.)

Proposition 2. *Let $a\in O$ and $a=2^b\neq 1$. If P belongs to the class $\mathcal{K}_a$ then P is expressible in the form* (3.8), *where* $Q(\alpha, x, n)\equiv$

[4] Ljapunow et al. [7; p. 12].

$(i)Q_1(\alpha, x, n, i)$, $Q_1 \in \mathcal{K}_b$ *and* Q *satisfies the condition* (3.9).

Proof is easily done. (Cf. Proof of Proposition 1.)

§ 4. In this section, we shall concern with functionals. A functional $F\langle\alpha, x\rangle$ means a mapping from $(N^N)^r \times N^s$ into N^N, where $r \geq 0$ and $s \geq 0$. Hence the value of $F\langle\alpha, x\rangle(y)$ for a natural number y is a natural number. If $r=0$ then it is an $(s+1)$-place number-theoretic function. Now we shall define the classes $\mathcal{C}_a$ and $\mathcal{L}_a (a \in O)$ of functionals $F\langle\alpha, x\rangle$ as follows:

For $a=1$, $\mathcal{C}_1$ is the class of all general recursive functionals $F\langle\alpha, x\rangle$.

For $a=b^* \neq 1$, $\mathcal{C}_a$ consists of all functionals $F\langle\alpha, x\rangle$ such that for some functional $G\langle\alpha, x, n\rangle$ contained in $\mathcal{C}_b$ the following (4.1) holds:

$$(4.1) \qquad F\langle\alpha, x\rangle = \lim_n G\langle\alpha, x, n\rangle ,$$

namely

$$(4.1') \qquad (\alpha)(x)(y)(Ek)(n)_{n\geq k}[\overline{F\langle\alpha, x\rangle}(y) = \overline{G\langle\alpha, x, n\rangle}(y)] .$$

For $a=3\cdot 5^b$, $\mathcal{C}_a$ consists of all functionals $F\langle\alpha, x\rangle$ such that there are functional $G\langle\alpha, x, n\rangle$ and general recursive functions ξ, η by which the following (4.2)–(4.4) are satisfied:

$$(4.2) \qquad \xi(n) <_0 a \text{ for all } n ,$$

$$(4.3) \qquad \text{For all } n, \alpha \quad \lambda xyG\langle\alpha, x, n\rangle(y) \leq_T [\eta(n)]\{H^{\alpha}_{\omega(\xi(n))}, \alpha\} ,$$

$$(4.4) \qquad F\langle\alpha, x\rangle = \lim_n G\langle\alpha, x, n\rangle .$$

Let $\{w_i\}$ be an effective enumeration of all sequence numbers without repetitions. Then for each $a \in O$, $\mathcal{L}_a$ is the class of all functionals $F\langle\alpha, x\rangle$ such that the predicate $\lambda\alpha xi[\overline{F\langle\alpha, x\rangle}(lh(w_i)) = w_i]$ is contained in the class $\mathfrak{M}_a$.

Lemma 8. *Let* $a \in O$. *If* $Fin(a)$, then $\mathcal{C}_a = \mathcal{L}_a$.

This is the theorem proved in [9], by Lemma 1 above. With respect to the other hierarchy, this fact was already obtained by Kondô [6].

Remark 4. Of course if $a, b \in O$ and $|a| = |b|$ then $\mathcal{L}_a = \mathcal{L}_b$. (Cf. Lemma 3 above.)

Lemma 9. *Let* $a \in O$ *and* $\overline{Fin}(a)$. *If* $F\langle\alpha, x\rangle$ *is contained in the class* $\mathcal{C}_a$, *then it is also contained in the class* $\mathcal{L}_{a^*}$. *That is,* $\mathcal{C}_a \subseteq \mathcal{L}_{a^*}$.

Proof. Case 1. $a=3\cdot 5^b$. Then there are a functional G and general recursive functions ξ, η such that (4.2)–(4.4) hold. By (4.4) we have

$$(4.5) \qquad \overline{F\langle\alpha, x\rangle}(lh(w_i)) = w_i \equiv \lim_n [\overline{G\langle\alpha, x, n\rangle}(lh(w_i)) = w_i] .$$

By (4.3) we can find a Gödel number e such that

$$\lambda nxyG\langle\alpha, x, n\rangle(y) \leq_T [e]\{H^{\alpha}_{\omega(a)}, \alpha\}$$

for all α. Hence the predicate $\overline{G\langle\alpha, x, n\rangle}(lh(w_i))=w_i$ is contained in $\mathfrak{M}_a$, namely G is in $\mathfrak{L}_a$. Consequently F belongs to the class $\mathfrak{L}_{a^*}$, by (4.5).

Case 2. $a=b^*$. Then for some functional G in $\mathfrak{C}_b$ (4.1) holds. By the hypothesis of induction G belongs to the class $\mathfrak{L}_{b^*}=\mathfrak{L}_a$. Therefore by (4.5) we have $F\in\mathfrak{L}_{a^*}$. (Q.E.D.)

Corresponding to [9; Lemmas 4 and 5] we can obtain the following two lemmas:

Lemma 10. *Let $a\in O$ and $\overline{Fin}(a)$. Under the hypothesis $\mathfrak{C}_a=\mathfrak{L}_{a^*}$: Let $F\langle\alpha, x\rangle$ be a functional such that for some $G\langle\alpha, x, k\rangle$ contained in $\mathfrak{C}_{a^*}$*

$$F\langle\alpha, x\rangle=\lim_k G\langle\alpha, x, k\rangle \quad \textit{recursively uniformly.} \tag{4.6}$$

This means that there exists a general recursive (monotone increasing) function $\theta(k)$ such that

$$(\alpha)(x)(k)(m)[m\geq\theta(k)\rightarrow\overline{G\langle\alpha, x, m\rangle}(k)=\overline{F\langle\alpha, x\rangle}(k)]\ . \tag{4.7}$$

Then F is contained in the class $\mathfrak{C}_{a^*}$.

Proof is similar to [9; Proof of Lemma 4.].

Lemma 11. *Let $a\in O$ and $\overline{Fin}(a)$. Under the hypothesis $\mathfrak{L}_{a^*}\subseteq\mathfrak{C}_a$: It holds $\mathfrak{L}_{a^{**}}\subseteq\mathfrak{C}_{a^*}$.*

Proof. Suppose $F\langle\alpha, x\rangle$ belongs to the class $\mathfrak{L}_{a^{**}}$. After the method of the proof of [9; Lemma 5], let β define as follows:

$$\beta(k, t, i)=\begin{cases}(w_i)_k\dot{-}1 & \text{if } t=lh(w_i)\ \&\ k<t\ ,\\ 0 & \text{otherwise.}\end{cases}$$

Then β is a recursive function. Since F belongs to $\mathfrak{L}_{a^{**}}$, the predicate (of α, x, i, t):

$$t=lh(w_i)\ \&\ \overline{F\langle\alpha, x\rangle}(t)=w_i$$

(denoted by $P(\alpha, x, i, t)$) is in the class $\mathfrak{M}_{a^{**}}$. Hence by Remark 3 P is contained in $(\mathfrak{M}_{a^*})^*$. Therefore, there exists a predicate V contained in $\mathfrak{M}_{a^*}$ such that

$$P(\alpha, x, i, t)\equiv\lim_n V(\alpha, x, i, t, n)\ . \tag{4.8}$$

Now we shall define two functionals G, J as follows:

$$G\langle\alpha, x, t\rangle(y)=\beta(y, t, i) \qquad \text{if } P(\alpha, x, i, t)\ , \tag{4.9}$$

and

$$J\langle\alpha, x, t, n\rangle(y)=\beta(y, t, (\mu j)_{j\leq n}V(\alpha, x, i, t, n))\ , \tag{4.10}$$

Since we have

(4.11) $$(t)(\alpha)(x)(E!\,i)P(\alpha, x, i, t)\ ,$$

the definition of G is correct. As V belongs to $\mathfrak{M}_{a^*}$,

$$\lambda\alpha xtnyz[J\langle\alpha, x, n\rangle(y)=z]$$

is also contained in $\mathfrak{M}_{a^*}$, and hence J belongs to the class $\mathfrak{L}_{a^*}$. By the hypothesis J also belong to $\mathcal{C}_a$. Using (4.8)–(4.11) it can be shown that

$$G\langle\alpha, x, t\rangle=\lim_n J\langle\alpha, x, t, n\rangle$$

and

$$(\alpha)(x)(k)(t)[t\geq k\rightarrow\overline{G\langle\alpha, x, t\rangle}(k)=\overline{F\langle\alpha, x\rangle}(k)]\ .$$

Applying Lemma 10 with $\theta(k)=k$ and with Lemma 9 we have $F\in\mathcal{C}_{a^*}$. Thus, Lemma 11 is proved.

Lemma 12. *Let $a\in O$ and $|a|$ be a limit number. Then $\mathfrak{L}_{a^*}\subseteq\mathcal{C}_a$.*

Proof. Let $F\langle\alpha, x\rangle$ be a functional contained in $\mathfrak{L}_{a^*}$. We use the notation in the proof of Lemma 11. Since the predicate $P(\alpha, x, i, t)$ belongs to the class $\mathfrak{M}_{a^*}=(\mathfrak{M}_a)^*$, by Lemma 7 P is in the class $\mathcal{K}_a$. So there are a predicate $V(\alpha, x, i, t, n)$ and general recursive functions ξ, η such that

(4.12) $$\xi(n)<_0 a \qquad \text{for all } n,$$

(4.13) $$\text{For all } n, \alpha \quad \lambda xit\,V(\alpha, x, i, t, n)\leq_T[\eta(n)]\{H^{\alpha}_{\omega(\xi(n))}, \alpha\}\ ,$$

(4.14) $$P(\alpha, x, i, t)\equiv\lim_n V(\alpha, x, i, n)\ .$$

Let G and J be the functionals defined in (4.9) and (4.10), respectively. Taking account of

$$V(\alpha, x, i, t, n)\equiv U(\mu y T^{H^{\alpha}_{u},\alpha}_{s+2}(\eta(n), x, i, t, y))=0\ ,$$

where $u=\omega(\xi(n))$, let q be a uniform Gödel number of

$$\lambda nxtk\beta(k, t, (\mu i)_{i\leq n}[U(\mu y T^{A,\alpha}_{s+2}(\eta(n), x, i, t, y))=0]$$

from A and α. Then it holds:

(4.15) $$\lambda xtkJ\langle\alpha, x, t, n\rangle(k)\leq_T[S^{\overbrace{1,1,1,\cdots,1}^{r}}_{s+2}(q, n)]\{H^{\alpha}_{\omega(\xi(n))}, \alpha\}$$

for all n, α. Further we shall define the functional J' as follows:

(4.16) $$J'\langle\alpha, x, n\rangle(k)=J\langle\alpha, x, k+1, n\rangle(k)\ .$$

Then by (4.15) and (4.16) we can find a general recursive function η' such that

(4.17) $$\lambda xkJ'\langle\alpha, x, n\rangle(k)\leq_T[\eta'(n)]\{H^{\alpha}_{\omega(\xi(n))}, \alpha\}$$

for all n, α. Since we can see

$$F\langle\alpha, x\rangle=\lim_n J'\langle\alpha, x, n\rangle\ ,$$

F is contained in the class $\mathcal{C}_a$ by (4.12) and (4.17). This completes the proof of Lemma 12.

Corollary 4. *Let $a \in O$ and $\overline{Fin}(a)$. Then $\mathfrak{L}_{a^*} \subseteq \mathcal{C}_a$.*

For, let $a=2^b$ and $\overline{Fin}(a)$. Then by Lemma 11 together with the hypothesis of induction $\mathfrak{L}_{b^*} \subseteq \mathcal{C}_b$, we have $\mathfrak{L}_{a^*} \subseteq \mathcal{C}_a$. If $a=3 \cdot 5^b$ then $\mathfrak{L}_{a^*} \subseteq \mathcal{C}_a$ by Lemma 12. (Q.E.D.)

Using Lemma 9 and Corollary 4 we have the

Theorem 2. *Let $a \in O$ and $\overline{Fin}(a)$. Then $\mathfrak{L}_{a^*} = \mathcal{C}_a$.*

By Remark 4 we have the

Corollary 5. *If $a, b \in O$ and $|a|=|b|$, then $\mathcal{C}_a = \mathcal{C}_b$.*

Remark 5. By Theorem 1 and Lemma 4 we may say as follows: Let $a \in O$. If $Fin(a)$, a functional $F\langle \alpha, x\rangle$ belongs to the class $\mathfrak{L}_a$ if and only if the predicate

$$[\overline{F\langle\alpha, x\rangle}(lh(w_i))=w_i] \tag{4.18}$$

is contained in the class $\mathcal{K}_a$. If $\overline{Fin}(a)$, F is in $\mathfrak{L}_{a^*}$ if and only if the predicate (4.18) belongs to the class $\mathcal{K}_a$.

Therefore if we define $\mathfrak{L}'_a$ as the class of functionals $F\langle \alpha, x\rangle$ such that the predicates (4.18) belong to the class $\mathcal{K}_a$, then Theorem 2 with Lemma 8 becomes the following proposition:

Proposition 3. *Let $a \in O$.* Then $\mathcal{C}_a = \mathfrak{L}'_a$.

This gives an effective version of the Lebesgue's theorem in the classical theory of Baire's functions [7; p. 97].

§ 5. In this section we shall obtain the separation theorems. A predicate $P(\alpha, x, n)$ is said to be multiply separable by a predicate $D(\alpha, x, n)$, if

$$\overline{(E\alpha, x)}(n)D(\alpha, x, n) \ \& \ (\alpha)(x)(n)[P(\alpha, x, n) \rightarrow D(\alpha, x, n)]\,. \tag{5.1}$$

First separation theorem. *Let $a \in O$ and $P(\alpha, x, n)$ be a given predicate satisfying the following condition:*

$$\overline{(E\alpha, x)}(n)P(\alpha, x, n)\,. \tag{5.2}$$

When $a=b^* \neq 1$, if

$$P(\alpha, x, n) \equiv (k)R(\alpha, x, n, k) \tag{5.3}$$

for some R in $\mathcal{K}_b$, then $P(\alpha, x, n)$ can be multiply separated by a predicate $D(\alpha, x, n)$ contained in the class $\mathcal{B}_a$. When $a=3 \cdot 5^b$, if there are a predicate R and general recursive functions ξ, η for which (5.3) *and the following* (5.4), (5.5) *hold:*

(5.4) $$\xi(k) <_0 a \quad \textit{for all } k.$$

(5.5) $$\textit{For all } k, \alpha \quad \lambda xnR(\alpha, x, n, k) \leq_T [\eta(k)]\{H^{\alpha}_{\omega(\xi(n))}, \alpha\},$$

then $P(\alpha, x, n)$ is multiply separable by a predicate $D(\alpha, x, n)$ contained in the class $\mathfrak{B}_a$.

Proof can be established straightforward after the method of Ljapunow et al. [7; p. 17]. To make sure, we shall prove it. Let $R'(\alpha, x, n, k) \equiv (m)_{m\leq k} R(\alpha, x, n, m)$, and define $Q(\alpha, x, k)$ as follows:

(5.6) $$Q(\alpha, x, k) \equiv (n)_{n\leq k} R'(\alpha, x, n, k).$$

Then we have

(5.7) $$P(\alpha, x, n) \equiv (k) R'(\alpha, x, n, k),$$

(5.8) $$R'(\alpha, x, n, k+1) \rightarrow R'(\alpha, x, n, k),$$

(5.9) $$Q(\alpha, x, k+1) \rightarrow Q(\alpha, x, k) \qquad \text{(by (5.8))}.$$

Hence by (5.2) and (5.9) we obtain

(5.10) $$\overline{(E\alpha, x)}(k) Q(\alpha, x, k).$$

Let $D(\alpha, x, n)$ be

$$(Ek)[\{R'(\alpha, x, n, 0). \vee . k>0 \ \&\ R'(\alpha, x, n, k) \ \&\ Q(\alpha, x, k \dot{-} 1)\} \\ \&\ \{\bar{R}'(\alpha, x, n, k) \vee \bar{Q}(\alpha, x, k)\}].$$

Since by (5.8) and (5.9)

$$R'(\alpha, x, n, k+1) \ \&\ Q(\alpha, x, k+1) \rightarrow R'(\alpha, x, n, k+1) \ \&\ Q(\alpha, x, k)$$

and

$$R'(\alpha, x, n, k+1) \ \&\ Q(\alpha, x, k) \rightarrow R'(\alpha, x, n, k) \ \&\ Q(\alpha, x, k),$$

we have

$$\bar{D}(\alpha, x, n) \equiv \bar{R}'(\alpha, x, n, 0) \vee (Ek)[R'(\alpha, x, n, k) \ \&\ Q(\alpha, x, k) \\ \&\ \{\bar{R}'(\alpha, x, n, k+1) \vee \bar{Q}(\alpha, x, k)\}],$$

by using (5.10). Now let R_0 and R_1 define as follows:

$$R_0(\alpha, x, n, k) \equiv \begin{cases} R'(\alpha, x, n, 0) \ \&\ \{\bar{R}'(\alpha, x, n, 0) \vee \bar{Q}(\alpha, x, 0)\} & \text{if } k=0, \\ R'(\alpha, x, n, k) \ \&\ Q(\alpha, x, k \dot{-} 1) \ \&\ \{\bar{R}'(\alpha, x, n, k) \vee \bar{Q}(\alpha, x, k)\} & \\ & \text{if } k>0, \end{cases}$$

$$R_1(\alpha, x, n, k) \equiv \begin{cases} \bar{R}'(\alpha, x, n, 0) & \text{if } k=0, \\ R'(\alpha, x, n, k) \ \&\ Q(\alpha, x, k) \ \&\ \{\bar{R}'(\alpha, x, n, k+1) \vee \bar{Q}(\alpha, x, k)\} & \\ & \text{if } k>0. \end{cases}$$

Then obviously we have

(5.11) $$D(\alpha, x, n) \equiv (Ek) R_0(\alpha, x, n, k) \equiv (k) \bar{R}_1(\alpha, x, n, k).$$

When $a=b^*\neq 1$, R belongs to the class $\mathcal{K}_b$. Hence it can be easily shown that R_i $(i=0, 1)$ are also contained in $\mathcal{K}_b$. When $a=3\cdot 5^b$, there are general recursive functions ξ, η such that (5.4), (5.5) hold. Hence for a suitable recursive function η'

$$\lambda xnR'(\alpha, x, n, k)\leq_T[\eta'(k)]\{H^{\alpha}_{\omega(\xi(k))}, \alpha\}$$

holds for all k, α. So we can readily proved that there exist general recursive functions $\pi(k), \eta_i(k)$ $(i=0, 1)$ such that the following (5.12), (5.13) hold for $i=0, 1$:

(5.12) $$\pi(k)<_0 a \qquad \text{for all } k,$$

(5.13) For all k, α $$\lambda xnR_i(\alpha, x, n, k)\leq_T[\eta_i(k)]\{H^{\alpha}_{\omega(\pi(k))}, \alpha\}\,.$$

Thus the predicate D belongs to the class $\mathcal{B}_a$ by (5.11). But by the method of proof of the corresponding classical theorem (see, [7; pp. 17–18]) we can obtain:

$$P(\alpha, x, n)\rightarrow D(\alpha, x, n)$$

and

$$\overline{(E\alpha, x)}(n)D(\alpha, x, n)\,.$$

Consequently. $P(\alpha, x, n)$ is multiply separable by the predicate $D(\alpha, x, n)$ contained in the class $\mathcal{B}_a$. (Q.E.D.)

Second separation theorem. *Let $a\in O$ and $P(\alpha, x, n)$ be a given predicate. When $a=b^*\neq 1$, if (5.3) holds for some predicate R in $\mathcal{K}_b$ then the predicate*

(5.14) $$V(\alpha, x, n)\equiv_{df}P(\alpha, x, n)\ \&\ (Ei)\bar{P}(\alpha, x, i)$$

is muliply separated by a predicate $D(\alpha, x, n)$ of the following form

(5.15) $$D(\alpha, x, n)\equiv(Ek)W(\alpha, x, n, k)$$

for some W contained in $\mathcal{K}_b$. When $a=3\cdot 5^b$, if there are R and general recursive functions ξ, η for which (5.3)–(5.5) *hold, then the predicate $V(\alpha, x, n)$ defined by* (5.14) *is multiply separated by a predicate $D(\alpha, x, n)$ satisfying the following condition:*

There are a predicate W and general recursive functions ξ', η' such that (5.15) *and the following* (5.16), (5.17) *hold:*

(5.16) $$\xi'(k)<_0 a \qquad \text{for all } k,$$

(5.17) For all k, α $$\lambda xnW(\alpha, x, k)\leq_T[\eta'(k)]\{H^{\alpha}_{\omega(\xi'(k))}, \alpha\}\,.$$

Proof. This theorem can also be proved straightforward along the classical proof. We follow after [7; p. 18]. Let $a\in O$ and $a=3\cdot 5^b$. We define $Q(\alpha, x, k)$ as in the proof of the preceding theorem, and put

$$W(\alpha, x, n, k) \equiv R'(\alpha, x, n, k) \ \& \ \bar{Q}(\alpha, x, k) ,$$
$$D(\alpha, x, n) \equiv (Ek) W(\alpha, x, n, k) .$$

Then it can be shown that

$$V(\alpha, x, n) \to D(\alpha, x, n) \quad \text{and} \quad (\overline{E\alpha, x})(n) D(\alpha, x, n) ,$$

where V is the predicate defined by (5.14). For the sake of the proof of the former, the equivalence: $(k)Q(\alpha, x, k) \equiv (n)P(\alpha, x, n)$ is used. Further we can find general recursive functions ξ', η' by which (5.16) and (5.17) are satisfied for the predicate W. This gives the proof for $|\alpha|$ limit. Proof of another case is omitted.

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